J Electroceram (2007) 19:427–431
DOI 10.1007/s10832-007-9033-6

Nonlinear elastic phenomena near the radial antiresonance frequency in piezoceramic discs

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Received: 6 March 2006 / Accepted: 8 September 2006 / Published online: 16 February 2007

Abstract The aim of this work is to study the nonlinear behaviour of a ceramic through the analysis of the radial mode at resonance and antiresonance frequencies. To obtain the nonlinear characterization from the impedance measurements, a correction of the current and impedance is proposed in order to obtain simple relationships between the motional impedance and the mean stress. The obtained relations are weakly dependent on frequency. This behaviour is also verified at resonance and at antiresonance. We use a nonlinear model where the elastic, dielectric and piezoelectric coefficients depend on the mean amplitude of the stress, which is more complete than the discrete elements models (RLC type), that are fulfilled only near the resonance.

Keywords Elastic nonlinearity · Antiresonance · Radial resonance

1 Introduction

When a nonlinear piezoelectric oscillator is excited by a high harmonic voltage, some new phenomena appear: there is a change in resonant frequency and in quality factor [1–3]; harmonic and subharmonic tones can be generated, and hysteresis appears. Some of these effects can be described by the change of the electrical impedance that occurs when the amplitude rises. In resonance, the oscillator can be placed under high mechanical stress, which not only alters its mechanical stiffness but also changes its piezoelectric and dielectric behaviour. The electrical impedance can be affected by all this changes.

The aim of this work is to study the nonlinear behaviour of a ceramic through the analysis of the radial mode at resonance and antiresonance frequencies. Radial resonance measurements are experimentally easy, but its theoretical treatment is rather complex. Empirically, it is known that the electrical impedance increases as a function of the vibration amplitude, and that the frequency dependence is not very significant if we are working near the resonance [2].

It can be seen [4] that the nonlinear behaviour of soft PZT ceramics shows a linear increment of the impedance with the current amplitude, while hard PZT shows an almost quadratic dependence. In order to better analyze the results, we must take only the motional component of the impedance, and relate it to the motional current, which is directly correlated with the internal stress. With that improvement, we observe a regular behaviour that even holds far away from resonance. We have shown that, with a suitable correction, such regularity can be observed even for amplitudes and frequencies that are able to produce antiresonance.

In this work, the experimental method used is described, the best way of analyzing the results is discussed, and it is demonstrated that this allows a simple description in a wide range of frequencies. By applying a suitable model, the radial resonator is analysed, and these results are theoretically justified. We use a model where the coefficients depend on the mean amplitude of the stress, which is more complete than the discrete elements model, but simpler than models where the coefficients depend on the position r and time t.

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2 Nonlinear analysis of radial resonator

In order to interpret the impedance increment as a function of the oscillation amplitude, it suffices to take a lumped element model. However, in order to make a comparison with the results obtained in other nonlinear experiences, this first model is not satisfactory, because it does not take into account the two-dimensional character of the vibration, and does not discriminate between nonlinear elastic, piezoelectric and dielectric effects.

Thus we take a model in which we assume the radial vibration of a piezoelectric disc, in a similar form as in a linear resonator, but assuming that the elastic coefficients depend on the stress value.

In spite of the spatial dependence of the stress, we will consider that the coefficients are independent of r, and they depend only on the mean value of the stress.

This model can work quite well near the fundamental mode of vibration, but it is not suitable for describing overtones, where one needs to consider the spatial dependence of the coefficients, and it is not able to describe harmonic generation, where the time evolution needs to be considered.

The advantage of this model is that it enables us to work with analytic solutions (Bessel equations), while the more complex models require numerical methods.

$$\begin{cases} T_r = c_{11}^P S_r + c_{12}^P S_\theta - e_{31}^P E_3 \\ T_\theta = c_{12}^P S_r + c_{11}^P S_\theta - e_{31}^P E_3 \\ D_3 = e_{31}^P S_r + e_{31}^P S_\theta - \varepsilon_{33}^P E_3 \end{cases} \tag{1}$$

By following IEEE Standards [5] we will take the next constitutive equations:

Those are linked to the d-form constitutive equations by the relations:

$$\begin{aligned} & c_{11}^P \cdot s_{11}^E \cdot \left(1 - \sigma^{P^2}\right) = 1, \\ & c_{12}^P = \sigma^P \cdot c_{11}^P; \text{ where } \quad \sigma^P = -s_{12}^E / s_{11}^E \\ & e_{31}^P = d_{31} \cdot \left(1 + \sigma^P\right) \cdot c_{11}^P, \\ & \varepsilon_{33}^T = \varepsilon_{33}^P + 2 e_{31}^P d_{31}. \end{aligned} \tag{2}$$

The differential equation is: $-\beta^2 \cdot u = \frac{\partial^2 u}{\partial^2 r} + \frac{1}{r}\frac{\partial u}{\partial r} - \frac{u}{r^2}$, where: $\beta^2 = \frac{\omega^2}{v^2} = \frac{\rho\omega^2}{c_{11}}$

And its solution is the Bessel function: $u = A \cdot J_1(x) \quad ; \quad x = \beta \cdot r$.

So, we can write the mechanical strain as:

$$S_\theta(r) = A\beta \frac{J_1(x)}{x} \quad ; \quad S_r(r) = A\beta\left(J_0(x) - \frac{J_1(x)}{x}\right). \tag{3}$$

The stress and the strain at each point of the disc are related to two quantities, which are both derived from the electric displacement D (see Fig. 3):

$$\begin{aligned} & D' = D_3 - \varepsilon_{33}^T E_3 = d_{31} \cdot (T_r + T_\theta), \quad < D' >= I'/j\omega a \\ & D'' = D_3 - \varepsilon_{33}^P E_3 = e_{31}^P \cdot (S_r + S_\theta), \quad < D'' >= I''/j\omega a \end{aligned} \tag{4}$$

If the values of both permittivities are known, the mean values of strain and stress are experimentally accessible, because E and D are known quantities, so they are related with V and I, directly measured.

We define the mean stress as $T = \langle T_r + T_\theta \rangle$ and we consider that the nonlinear coefficients are functions of T but not of r:

$$\begin{cases} c_{11}^P = c_{11_0}^P (1 - f_{NL}(T)) \\ \sigma^P = \sigma_0^P (1 - h_{NL}(T)) \\ e_{31}^P = e_{31_0}^P (1 + e_{NL}(T)) \\ \varepsilon_{33}^P = \varepsilon_{33_0}^P (1 + \varepsilon_{NL}(T)) \end{cases} \tag{5}$$

These nonlinear functions, for example $f_{NL}(T)$ can be linear or quadratic, depending on the material. Then, two models could be considered:

- Rayleigh model (valid for soft ceramics), if they are linear functions of the absolute value of T:

$$f_{NL}(T) = \xi_{NL} \cdot |T|. \tag{6}$$

- Quadratic model (valid for hard ceramics), if they are quadratic functions of T:

$$f_{NL}(T) = \chi_{NL} \cdot T^2. \tag{7}$$

Note that a remarkable simplification has been made here, since a stress in direction 1 is not expected to affect the coefficients c_{11}^P and c_{22}^P at the same rate.

From the linear analysis of the disc vibration, we obtain that the total electrical admittance is:

$$Y = \frac{j\omega a}{t}\frac{\langle D \rangle}{E} = j\omega C^P \cdot \left(\frac{2e_{31}^{P^2}}{c_{11}^P \cdot \varepsilon_{33}^P}\frac{S_\theta(R)}{(S_r(R) + \sigma^P S_\theta(R))} + 1\right) \quad ; \quad C^P = \frac{a}{t}\varepsilon_{33}^P. \tag{8}$$

We can model the system as an equivalent circuit with two parallel branches, whose admittances are Y'' and Y^P. The admittance of the electrical branch is $Y^P = j\omega C^P$, so in the motional branch we have:

$$Y'' = Y - Y^P = j\omega C^P \cdot \left(\frac{2e_{31}^{P^2}}{c_{11}^P \cdot \varepsilon_{33}^P}\frac{S_\theta(R)}{(S_r(R) + \sigma^P S_\theta(R))}\right). \tag{9}$$

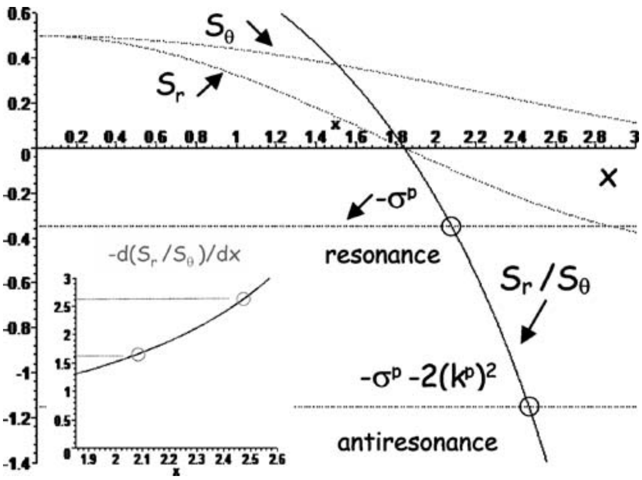

Fig. 1 Dependence of S_r/S_θ with x=ω*R/v*. This function determines the resonance and antiresonance conditions. In the inset, the plot of its slope is shown

Then, the impedance Z'' can be obtained:

$$Z'' = \frac{t}{j\omega a} \cdot \frac{c_{11}^P}{2e_{31}^{P2}} \cdot \left(\frac{S_r(x)}{S_\theta(x)} + \sigma^P \right) \quad ; \quad x = \beta R = \frac{\omega}{\sqrt{c_{11}^P}} R\sqrt{\rho}. \tag{10}$$

This equation enables us to evaluate the effect on Z'' induced by the stress: a nonlinear decrease of stiffness c_{11}^P, produced by a rise of the mean value of T (Eq. 5), will increase β, and x (Eq. 10). As x rises with frequency, an amplitude increase will have an effect similar to a frequency increase. If the coefficients do not depend on r, the function S_r/S_θ can be expressed by using the Bessel functions: $\frac{S_r(x)}{S_\theta(x)} = \frac{xJ_0}{J_1} - 1 = \mathbf{J}_1(x) - 1$ as can be seen in Fig. 1.

From Eq. 10, at the disc border the value of S_r/S_θ is equal to $-\sigma^P$ at resonance, while at antiresonance it is $-\sigma^P - 2e_{31}^{p^2} \Big/ \left(c_{11}^p \varepsilon_{33}^p\right)$.

The dependence of the impedance on the current amplitude $Z''(I')$ will depend on the slope of the function $S_r/S_\theta(x)$, which evolves smoothly in all the frequency range between resonance f_r and antiresonance f_a, so the nonlinear effect on Z'' is similar in all this range (as we can see in Figs. 2c and 3c), and only small quantitative differences can be observed in all the range, as can be clearly seen by plotting the slope of the previous equation (Fig. 1).

3 Experimental system

A sinusoidal voltage is applied to the disc at frequencies in the radial resonance-antiresonance range. It is supported at two points on its axis, where there are the electric contacts, allowing a free radial vibration. The measurements of the voltage *V(t)*, the current *I(t)* and the velocity in the disc border *v(R,t)* (by a laser vibrometer) are taken through a four channel oscilloscope and controlled by a computer. From these data the complex impedance Z as well as the phase between velocity and current are calculated.

A burst signal excitation is used to avoid the overheating of samples. The measurements are carried out in the steady-state zone, after a number of sufficient previous cycles (which is function of the resonator quality factor), to ensure that the transient zone is finished. In this steady-state interval a large number of measurement cycles is used for greater accuracy.

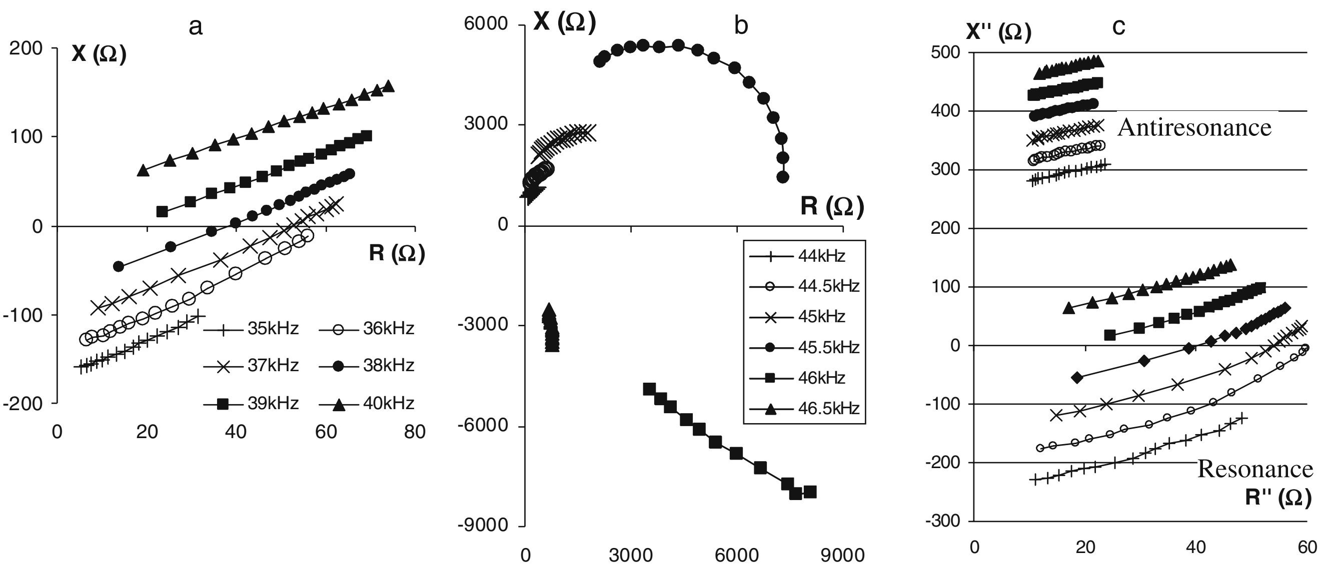

Fig. 2 Impedance plane *X(R)* measurements for a PXE5 disc (*R*=2,5 cm, *t*=2 mm) at constant frequency and sweep amplitudes. **a** Near resonance frequency; **b** near antiresonance; **c** the same points, in the motional *X''(R'')* plane

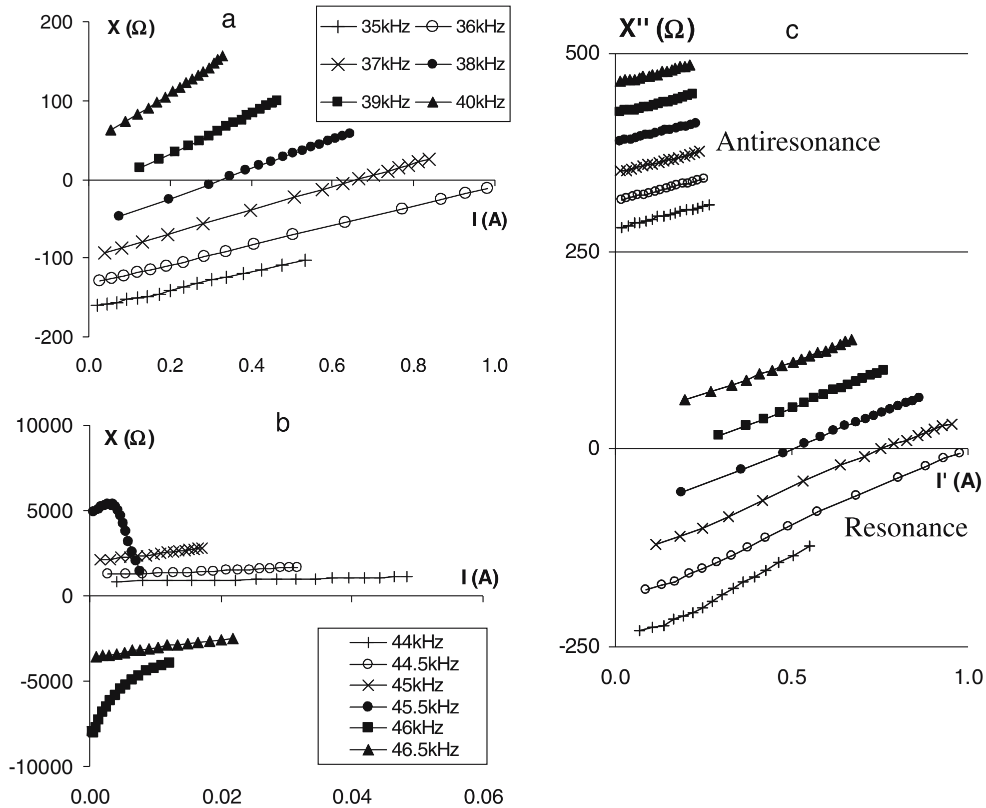

Fig. 3 Reactance versus current *X(I)* for the same measurements points: **a** near resonance frequency; **b** near antiresonance; **c** the modified *X″(I′)* representation at resonance and antiresonance

For a constant frequency, the signal is applied at increasing amplitudes in order to obtain the dependence of the impedance versus the amplitude. The variation $\Delta Z(I)$ is shown [2] to be weakly dependent on frequency: this dependence is verified by repeating the measurements at other close frequencies.

The complete process is also carried out at frequencies in the resonanace and antiresonance zones. Near the resonance, it is better to insert a serial resistor to prevent, or minimize, the hysteresis phenomenon, because this set-up is similar to a current generator, and prevents hysteresis [1].

For the data treatment it is necessary to know the capacitances C^T and C^p. C^T is the low frequency capacitance and C^p is the capacitance measured in the interval between the radial and thickness resonances, corresponding to ε^P_{33}.

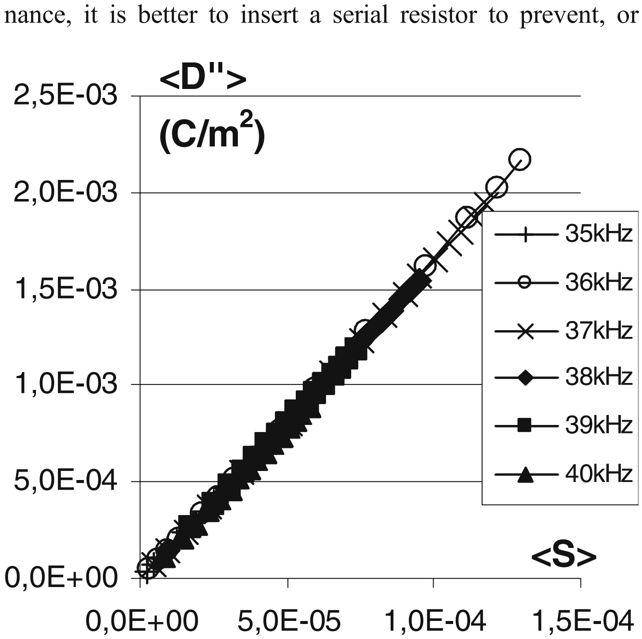

Fig. 4 Modified mean electric displacement <*D″*> versus the mean strain <*S*>

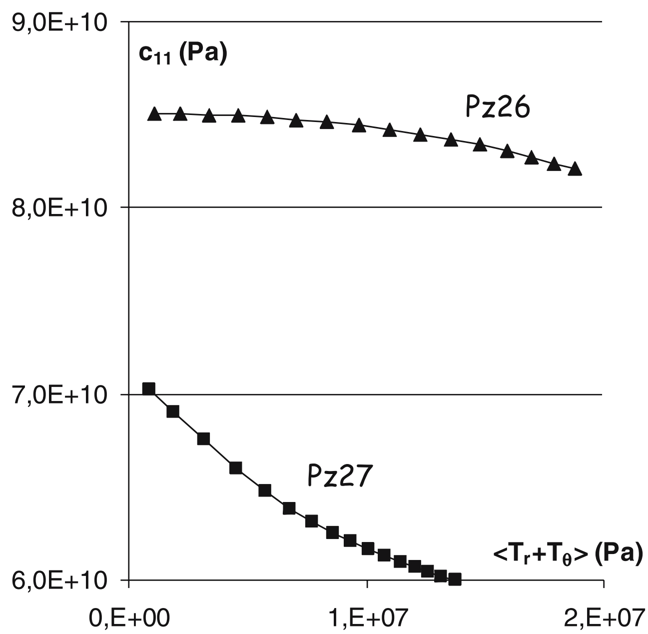

Fig. 5 Stiffness against the mean stress: soft Pz27 and hard Pz26 piezoceramics

The value I', which is proportional to the mean stress $\langle T \rangle$ (Eq. 4), can be computed through the admittance Y_0 value:

$$Y_0 = j\omega C^T, \quad 1/Z' = 1/Z - Y_0, \quad I' = V/Z'. \tag{11}$$

The motional impedance Z'' and reactance X'' are obtained from the capacitance C^p previously mentioned:

$$Y^P = j\omega C^P = j\omega \varepsilon_{33}^P a/t, \quad 1/Z'' = 1/Z - Y^P, \quad Z'' = R'' + jX'', \tag{12}$$

where a is the disc area and t the thickness.

From these magnitudes, the relation $X''(I')$ is obtained, which is approximately a straight line, with a slope that is weakly dependent on frequency in all the resonance-antiresonance frequency interval.

Finally, from the disc border velocity $v(R,t)$, the mean strain $\langle S \rangle = 2v(R)/j\omega R$ is calculated, and the relation between the modified electric displacement $\langle D'' \rangle = I''/j\omega a$ and $\langle S \rangle$ is obtained. From this relation the piezoelectric coefficient e_{31}^P can be calculated (Eq. 4).

4 Experimental results

When the amplitude level increases, the representation in the impedance plane $X(R)$ shows the experimental points in a straight line for each frequency (Fig. 2a): the reactance increases (and so, the stiffness c_{11}^P decreases) as well as the resistence (increase of losses) with the excitation amplitude. Similar representation can be seen from the reactance versus the current $X(I)$ (Fig. 3a). However, these functions are no longer regular when the frequency is far away from the resonance (Figs. 2b, 3b), where it is possible to observe the shift from resonance to the antiresonance by only increasing the level amplitude, at constant frequency.

However, when the functions $X''(R'')$ (Fig. 2c), or $X''(I')$ (Fig. 3c) are represented they show a regular behaviour both at resonance and at antiresonance. Due to this regularity it is possible to conclude that, in the studied frequency range, it is verified that the increases in motional reactance and stiffness are fundamentally dependent on the mean stress $\langle T \rangle$, and they are almost independent of the frequency.This fact justifies the measurement method used, because the parallelism of the experimental curves $X''(I')$, makes the multiple frequency sweeps unnecessary, since all the information can be obtained from a single amplitude sweep.

Figure 4 shows the relation between the modified electric displacement $\langle D'' \rangle$ and the mean strain $\langle S \rangle = \langle S_r + S_\theta \rangle$. It is verified that the piezoelectric coefficient e_{31}^P relating both magnitudes (Eq. 4) is almost constant.

If we suppose that there is a stress dependence of c_{11}^P, and that the Poisson coefficient remains constant, this model allows us to relate the actual stiffness value with the mean stress value. For each frequency and amplitude, there is an equivalent frequency ω_{eq} at which its impedance is the same as if its behaviour were linear. In order to find that frequency, we take the relation $X''(I')$ and find its limit value (linear) X_0'' at a null amplitude, as well as its dependence with the frequency $\mathrm{d}X_0''/\mathrm{d}\omega$. At this point, it is possible to obtain the equivalent frequency $\omega_{eq} = \omega + (X'' - X_0'')/(\mathrm{d}X_0''/\mathrm{d}\omega)$, and evaluate the stiffness (Eq. 10).

In Fig. 5 we show the result of applying this procedure to two different ceramics, soft and hard, by showing the two types of nonlinear behaviour expected (Eq. 5): a quasi-linear dependence in soft Pz27 ceramic (Eq. 6) and a quadratic dependence (Eq. 7) for hard Pz26 ceramic.

5 Conclusion

A measurement method and data treatment is proposed, that will enable the analysis of the dependence of the elastic coefficient with the stress by using the increase of electrical impedance.

The nonlinear description obtained is simple, and it is valid in all the range between resonance and antiresonance frequencies.

The nonlinear behaviour, that is shown by a regular increase of motional reactance with stress amplitude, can be explained by the theoretical model proposed, and the nonlinear coefficients can be obtained by assuming some assumptions: the use of mean values of stress instead of radial dependence and the fact that the Poisson coefficient remains constant.

At antiresonance, where the electric field is not null, it is necessary to take into account the dielectric and piezoelectric nonlinearities.

Unfortunately, some problems still remain, and it is not possible to clearly distinguish between all the nonlinear effects. In order to attain it, it will be necessary to use more sophisticated and complete models, which enable to obtain information from the behaviour at the overtones or from harmonic generation.

Acknowledgments This work is supported by the Spanish MEC (project MAT2004-01341) and the European Network POLECER (G5RT.CT-2001-05024).

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